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T1 ________	**50099**	F1 ________
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T3 ________	Problem Chosen	F3 ________
T4 ________	**A**	F4 ________

2016
Mathematical Contest in Modeling (MCM/ICM) Summary Sheet

Bath Model Based on Heat Transfer Model and Analytic Hierarchy Process

Summary

Taking a bath never fails to fascinate human beings, owing to the fact it brings people cleanliness and relaxation. Lots of people go into raptures at the mere mention of bathing themselves in hot water. However, the bath always gets cooler and people are obliged to add hot water, which brings trouble. Hence what we have done is to solve such problem.

Based on Heat Transfer model and Analytical Hierarchy Process (AHP), we build a model to work out a practical strategy to maintain temperature throughout the bath water. First we list all the factors making for the change of the water temperature, such as the temperature of the air and the running hot water, the motion of the person and the material of the tub and so on.

Then we analyze the process of the thermal transfer among water, person, tub and air. Considering the rate and temperature of the hot water flow as the variables, we simulate the variation of the water temperature. We vary the variables and record the statistics. By defining two parameters—comfort index and thrift index, we are able to evaluate the performance of the strategy under different values of the variables.

At last we further discuss the effects caused by other factors and come up with a strategy applicable to a medium height and medium weight person who likes to bath in water of about 40 degrees centigrade in a ceramic bathtub. In other situations, the person can slightly adjust the strategy according to our further instructions.

Keywords: Heat Transfer Model; Analytical Hierarchy Process

Contents

1 Introduction

1.1 Analysis of the problem

1.1.1 Why it is difficult to maintain the temperature?

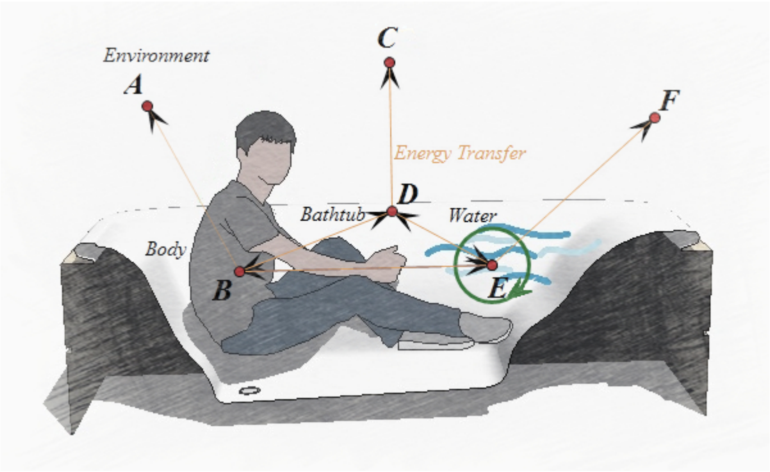


Figure 1: Heat Transfer during a Bath

- As is shown in the above figure, the heat or energy transfer makes for the variation of the water temperature.

- The temperature of the air, the host, and the water is always different, so that energy exchange will constantly happens among these objects. Thus, the temperature of water will frequently change, and will be hard to be stable.

- The water has Liquidity and the location and the speed of water molecules is unpredictable by the uncertainty principle. And with the host's movement, the flowing route of water is always changing randomly. So the mixture of water molecules and their energy exchange happens randomly all the time.Thus the temperature of water is varying all the time.

- There is a large space span between the faucet and the whole bathtub of water. So it's hard for the energy to spread out. In another word, the control ability of temperature from the tap is limited. Thus it's hard to control the temperature.

- Many variables don't have accurate value. For example, it's hard for the host to measure all the parameter we need. So the strategy is based on universal information or model deduction. But when the strategy was applied to the practical circumstance, those parameter is unpredictable, like Specific heat Capacity, thermal conductivity, the host's physical and emotional condition, etc. In another word, absolute prediction need absolutely large data and information, but that's hard. So, it's hard for the water to be evenly maintained.

1.2 Background and Tasks

Unlike spa, the bath always gets cooler and people have to add some hot water from the faucet to reheat the bathing water. Some people tend to add lots of hot water causing a waste of water resource and energy while others add too little to keep the bath temperature moderate. When taking a bath, it is impossible for us to measure the temperature all the time. Hence a practical and simple strategy to achieve the balance between thrift and comfort is demanded.

Based on our analysis, remain the hot water running after the tub is filled with water is the most simple way to achieve our goal. But how can we know the best velocity and temperature of water flow? How to adjust our strategy to the changing weather? What to do if our bathtub is made of other materials? People's motion during the bath also causes variation of heat. How to cope with those problems? Thus, our tasks are explicit:

- Explain why it is so difficult to get an evenly maintained temperature throughout the bath water.
- Set value for the variables (the velocity and the temperature of the water flow) and establish models to simulate the variation of water temperature.
- Propose models to evaluate the performance of our strategy. By varying the value of the variables, we find out the proper temperature and velocity of the running hot water.
- Extend the models to people of different height and weight. Extend the models to people who live in areas of different climates and use tubs of different types. Then test and analyze the performance of our strategy.
- Come up with the final strategy and explain it with non-technical explanation for users of the bathtub.

Based on the Heat Transfer Model, we establish a model to simulate the variation of the water temperature under different conditions. That model is divided into 4 submodels: the temperature field model, the inflow model, the outflow model and the heat transfer model. The temperature field model separates the bathing water into numerous parts to simulate the variation of water in an accurate way. Based on the temperature field model, the inflow and outflow model depicts the flow of water and the simulation of water temperature is theoretically supported by formulas in the heat transfer model.

1.3 Terminology

- **Comfort Index** : an index to evaluate the comfort of the bath, determined by the accumulation of number of water cubes whose temperature is comfortable.
- **Strategy or Method** : a way of keeping the hot water flowing at a specific rate and temperature which is determined by two variables.
- **Convection**: a kind of concerted, collective movement of groups or aggregates of molecules within fluids (e.g., liquids, gases) and rheids, through advection or through diffusion or as a combination of both of them.[1]

- **Conduction(heat)** : the transfer of internal energy by microscopic diffusion and collisions of particles or quasi-particles within a body or between contiguous bodies. [2]

1.4 Notation

Symbol	Meaning
Q_0/T_0	inflow rate/temperature
$T_b/T_m/T_a$	temperature of bathtub/human/air
$h_0/l_0/w_0$	height/length/width of bathtub
$h_m/l_m/w_m$	height/length/width of the cuboid representing human body
l_x	length of water cube
$nh/nl/nw$	number of cubes to cover bathtub height/length/width
T	temperature field function T=f(x,y,z,t)
T_1/T_2	temperature of adjacent cubes
ΔT	temperature change
n	normal vector of isothermal surface
Δt_0	our model updates per Δt_0 time
ΔV_0	water inflow per Δt_0 time
N_0	water cube inflow per Δt_0 time
Φ	heat transfer power
c/ρ	specific heat/density of water
λ	heat transfer coefficient of water
λ_x	action pattern coefficient
h	convection heat transfer coefficient between water and bathtub
C_i	the comfort index of Method i
V_i/E_i	the volume of water / the energy consumed in Method i
ΔQ	heat transfer amount

Table 1: Notation

1.5 Assumptions

- The hot water trickle falls on the middle point of the upper-left width of bathtub.
- The overflow drain is on the middle part of the upper-right width of bathtub.
- The tub is made of ceramic and is shaped like a cuboid.
- The bath lasts 30 minutes.
- The air temperature is fixed with 24 degrees centigrade and the moderate water temperature of the bath is 40 degrees centigrade.
- The temperature of water coming from the faucet varies from 25 degrees centigrade to 70 degrees centigrade.

2 Heat Transfer Models

2.1 Temperature Field Model

In consideration of the difference of temperature of different position in the bathtub, we use water cubes to simulate the temperature field in bathtub. In our simulation, we divide the bathtub into $l_x * l_x * l_x$ size cubes with a property of central point temperature($l_x = 10cm$, for example). Let $nl = \frac{l_0}{l_x}, nr = \frac{w_0}{l_x}, nh = \frac{h_0}{l_x}$, then the bathtub is divided into $nl*nr*nh$ water cubes. Also we can use coordinates to describe these cubes. In our model, (1,1,1) is upper left while (nl,nw,nh) is lower right. Note that despite of the contunity of temperature, temperatures in a small enough cube are of little difference. Thus during calculation process we can use the temperature of central point to represent the temperature of other points in a cube. For the reason above, the temperature field of the bathtub can be described by temperatures of a set of central points. This means that we only need to consider heat tranfer process among adjacent cubes.

We discuss the basic processes for the temperature field model:

- Inflow and Outflow Process:Simulating water inflow by generating new cubes with constant temperature in inflow-area. And simulating water overflow by removing cubes which is around the overflow drain.
- Heat Transfer Process:There are mainly three parts in heat transfer process, which are water-water heat transfer, water-air heat transfer and water-bathtub heat transfer.

Specific rules will be set in inflow model, overflow model and heattransfer model to simulate the hot bath process.

2.2 Inflow and Overflow Model

The inflow model simulates the hot water inflow process. As is assumed, hot water trickle falls around the upper-left width. We can calculate water inflow per Δt_0 time by $\Delta V_0 = Q\Delta t_0$. That is, $N_0 = \frac{\Delta V_0}{l_x * l_x * l_x}$ water cubes with temperature T_0. Meanwhile N_0 water cubes overflow from the overflow drain. To simulate this process, we follow 3 steps:

- delete cube $(nl, \frac{nw-N_0}{2}, 1)..(nl, \frac{nw+N_0}{2}, 1)$
- substitute cube $(i-1, \frac{nw-N_0}{2}, 1)..(i-1, \frac{nw+N_0}{2}, 1)$ for cube $(i, \frac{nw-N_0}{2}, 1)..(i, \frac{nw+N_0}{2}, 1)$ $i = 2..nl$
- generating new cube $(1, \frac{nw-N_0}{2}, 1)..(1, \frac{nw+N_0}{2}, 1)$ with temperature T_0

2.3 Heat Transfer Model

The heat transfer model simulates the temperature change via physics laws,which contains water-water heat transfer, water-air heat transfer and water-bathtub heat transfer.

2.3.1 Water-water Heat Transfer

Our water-water heat transfer model focuses on conduction. In our model, conduction only happens between adjacent water cubes. And we assume that temperature between the two central point follows linear relation. By this assumption, we keep the continuity of temperature. To calculate heat transfer, we use Fourier's Law:

$$\Phi = -\lambda A \frac{\partial T}{\partial n}$$

In our model, we let the interface of two cubes to be the isothermal surface for heat to transfer. In this sense, we have

$$A = l_x^2, \frac{\partial T}{\partial n} = \frac{T_1 - T_2}{l_x}$$

$$\Delta Q = \Phi \Delta t_0, \Delta T = \frac{\Delta Q}{c \rho l_x^3}$$

By formula above we are able to calculate the water-water heat transfer.

2.3.2 Water-Bathtub Heat Transfer

Water-bathtub heat transfer is mainly through convection. During the process we assume that bathtub temperature T_b is a constant. Newton's law of cooling is widely used in scientific literatures to solve convection heat transfer between liquid and solid:

$$\Phi = Ah(T_1 - T_b)$$

Similarly we have

$$A = l_x^2$$

$$\Delta Q = \Phi \Delta t_0, \Delta T = \frac{\Delta Q}{c \rho l_x^3}$$

By the method above we are able to calculate the water-bathtub heat transfer.

2.3.3 Water-air Heat Transfer

Water-air heat transfer contains processes of conduction, convection and radiation. In order to simulate the process more precisely, we apply function fitting method to measure water-air heat transfer.

2.4 Human Model

2.4.1 Human Shape

In common sense, people take bath with their head outside water and other part inside. To measure how comfort the bath is, we only need to consider the underwater part of human body. Simply, we use a $l_m * w_m * h_m$ rectangle to represent underwater bodypart. When in bathtub, most people will take a letter L posture. So in our model, we use a rectangle against profile with length h_0 and a rectangle against underside with length $h_m - h_0$ to describe the shape of human. Similarly, these rectangles are divided into small cubes with temperature.

2.4.2 Human Movement

Obviously, human movement may accelerate the heat transfer in the bathtub. And among the different heat transfer processes in our models, human movement has direct influence only on water-water process. Influence on other processes is passes via water-water process. In our opinion, the frequency and scope of human depend on his/her action pattern. Some people will stay there without much movement, while some will constantly stir the water. In this sense, we use a action pattern coefficient λ_x to describe how people with a certain action pattern will influence heat transfer process in the bathtub by equation

$$\lambda'_w = \lambda_x * \lambda_w$$

There are 3 kind of action patterns in our model, whose coefficients are shown below.

Pattern	Silent	Normal	Active
λ_x	1	1.2	1.7

Table 2: Action pattern coefficients of different action patterns

3 AHP Model

We classify all the strategies into 3 types:

- high rate and low temperature
- medium rate and medium temperature
- low rate and high temperature

3.1 Establish a Hierarchical Model

3.1.1 Factors affecting evaluation significantly

Comfort index

If you want to enjoy a cozy bath, the water temperature should not be too high or too low. For one thing, the temperature ought to be slightly higher than people's body's, for the evaporation of water takes away heat. For another, water hotter than 40 °C may well does harm to our cardiovascular system. We write the number of cubes contacting with person whose temperature is comfortable as **N1** and write the number of cubes contacting with person as **N2** . Thus, the comfort index is determined by the accumulation of number of water cubes whose temperature is comfortable:

$$C_i = \int \frac{N1}{N2} \Delta t$$

Thrift index

When taking a bath, we inevitably consume water resource and energy. Those are important factors reflecting how much natural resource you sacrifice for your comfort. Thus, the thrift index is determined by the quantity of water and heat used in the bath.

3.1.2 Hierarchy Figure

Through above analysis, hierarchy figure is shown below.

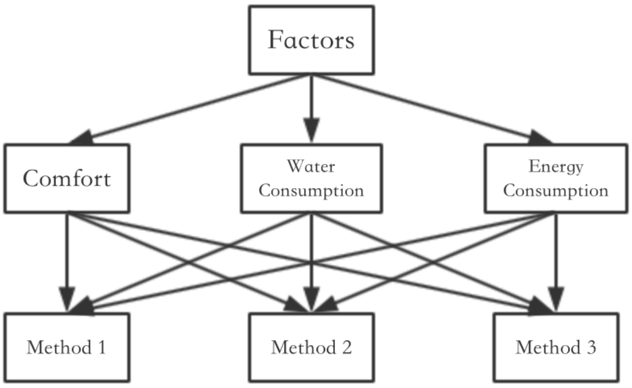


Figure 2: Hierarchy Figure

In **Hierarchy Figure** , Method 1 refers to the method which use water flow of low rate and high temperature. Method 2 refers to medium rate and medium temperature. Method 3 refers to high rate and low temperature.

3.2 Structure Comparison Matrix

We structure comparison matrix by Comparison Method of 1-9.

Factor	Comfort	Water Consumption	Heat Consumption
Comfort	1	1.557	1.557
Water Consumption	1/1.557	1	1
Heat Consumption	1/1.557	1	1

Table 3: Pairwise comparison matrix of hierarchy I-II

Comfort	Method 1	Method 2	Method 3
Method 1	1	$\frac{C_1}{C_2}$	$\frac{C_1}{C_3}$
Method 2	$\frac{C_2}{C_1}$	1	$\frac{C_2}{C_3}$
Method 3	$\frac{C_3}{C_1}$	$\frac{C_3}{C_2}$	1

Table 4: Pairwise comparison matrix of hierarchy II-III

Water Consumption	Method 1	Method 2	Method 3
Method 1	1	$\frac{V_1}{V_2}$	$\frac{V_1}{V_3}$
Method 2	$\frac{V_2}{V_1}$	1	$\frac{V_2}{V_3}$
Method 3	$\frac{V_3}{V_1}$	$\frac{V_3}{V_2}$	1

Table 5: Pairwise comparison matrix of hierarchy II-III

Energy Consumption	Method 1	Method 2	Method 3
Method 1	1	$\frac{E_1}{E_2}$	$\frac{E_1}{E_3}$
Method 2	$\frac{E_2}{E_1}$	1	$\frac{E_2}{E_3}$
Method 3	$\frac{E_3}{E_1}$	$\frac{E_3}{E_2}$	1

Table 6: Pairwise comparison matrix of hierarchy II-III

Through calculating, we can know that the max eigenvalue of pairwise comparison matrix of hierarchy I-II, II-III is $\lambda = 3.0$.

3.3 Select Method

The evaluation of each method is shown below.

Method Number	Evaluation
1	$\sum weight\_of\_factor * weight\_of\_method\_1\_in\_the\_factor$
2	$\sum weight\_of\_factor * weight\_of\_method\_2\_in\_the\_factor$
3	$\sum weight\_of\_factor * weight\_of\_method\_3\_in\_the\_factor$

Table 7: The Evaluation of each method

4 Model Implementation in Matlab

4.1 Environment Variables

4.1.1 Bathtub Size

Pattern	Small	Medium	Big
$Size$	1.2*0.7*0.3	1.4*0.8*0.4	1.6*0.9*0.5

Table 8: Different Bathtub Size

4.1.2 Human Shape

Pattern	Thin	Medium	Fat
w_m/m	0.1	0.2	0.3

Table 9: Degree of Obesity

Pattern	Short	Medium	Tall
w_m/m	0.1	0.2	0.3

Table 10: Human Height

We use a combination of degree of obesity and human height to describe human shape.

4.1.3 Person Temperature

Pattern	Low	Medium	High
Temperature/$^\circ C$	36	36.5	37

Table 11: Different Human Temperature

4.2 Implementation

4.2.1 Fitting of the water-air heat transfer formula

$$\Delta Q = \frac{c * mass \Delta T \Delta t0 * e^{\frac{\alpha}{kV+b}}}{kV + b}$$

Where k,v, b refer to constants.

4.2.2 The variation of the instant comfort index

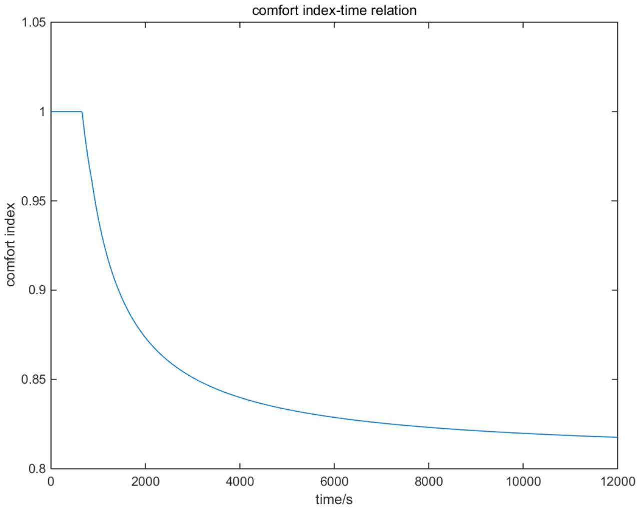


Figure 3: The instant comfort index under the condition that the water temperature is 42 degrees centigrade and the rate is 20L/min.

Based on the simulation of comfort index and calculation of thrift index, we can work out the evaluation of each method.

4.2.3 Evaluation of comfort index

By varying the variables(size of the bathtub, water rate and water temperature), we get the figure reflecting the comfort index of each strategy.

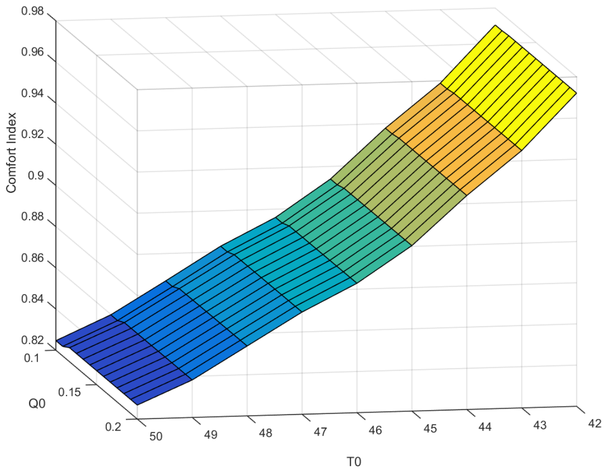


Figure 4: The comfort index in different situations in the middle size tub

4.2.4 Evaluation of each strategy

Finally, we get the figure reflecting the evaluation of each strategy.

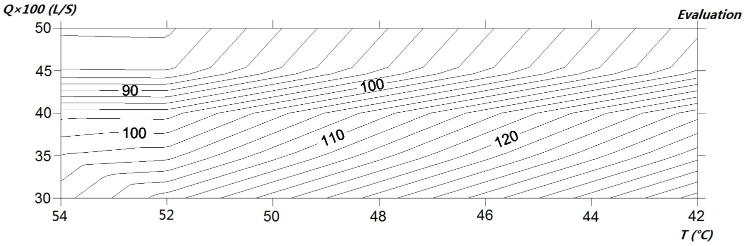


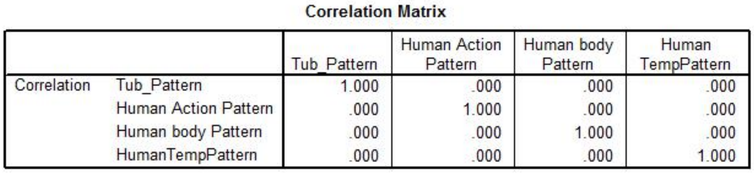
Figure 5: The evaluation in different situations in the middle size tub

5 One-way Analysis of Variance

5.1 PCA

In order to determine whether the response variableâĂŹs mean value has a significant difference under different initial variables or processing method, we employ the One-way analysis of variance to test our modelâĂŹs result.

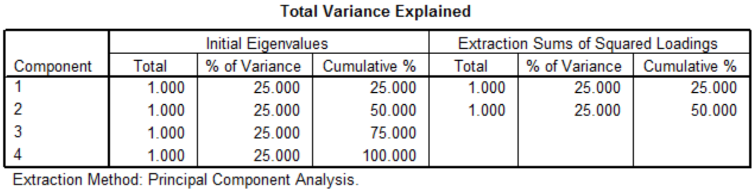
So, in the first step, we use the Principal Component Analysis to choose the principal component of environmental variables.



Correlation Matrix

		Tub_Pattern	Human Action Pattern	Human body Pattern	Human TempPattern
Correlation	Tub_Pattern	1.000	.000	.000	.000
	Human Action Pattern	.000	1.000	.000	.000
	Human body Pattern	.000	.000	1.000	.000
	HumanTempPattern	.000	.000	.000	1.000

Figure 6: The correlation analysis among indexes

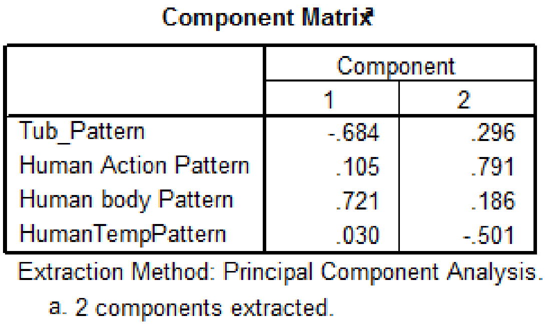


Total Variance Explained

Component	Initial Eigenvalues			Extraction Sums of Squared Loadings		
	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	1.000	25.000	25.000	1.000	25.000	25.000
2	1.000	25.000	50.000	1.000	25.000	50.000
3	1.000	25.000	75.000			
4	1.000	25.000	100.000			

Extraction Method: Principal Component Analysis.

Figure 7: The output of PCA analysis, total variance explained



Component Matrix[a]

	Component	
	1	2
Tub_Pattern	-.684	.296
Human Action Pattern	.105	.791
Human body Pattern	.721	.186
HumanTempPattern	.030	-.501

Extraction Method: Principal Component Analysis.

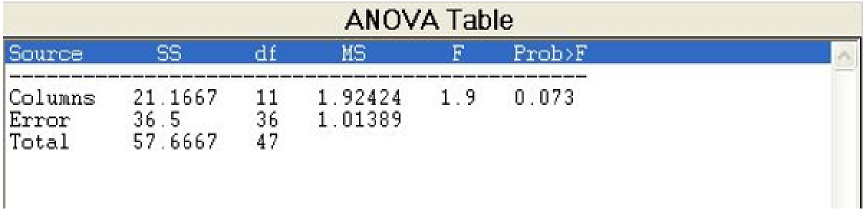
a. 2 components extracted.

Figure 8: The output of PCA analysis, component matrix

5.2 Result Analysis

By Matlab, we can get the information of analysis of variance:

In our test, we set Q0 and T0 versus the Tub_Pattern, Human_Action_Pattern, Human_body_Pattern, and Human_Temperature_Pattern.



ANOVA Table

```
Source      SS      df      MS        F     Prob>F
---------------------------------------------------
Columns   21.1667   11   1.92424    1.9    0.073
Error     36.5      36   1.01389
Total     57.6667   47
```

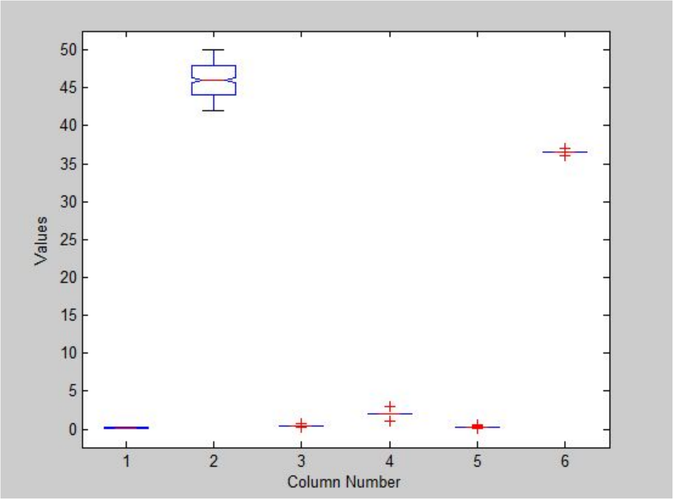
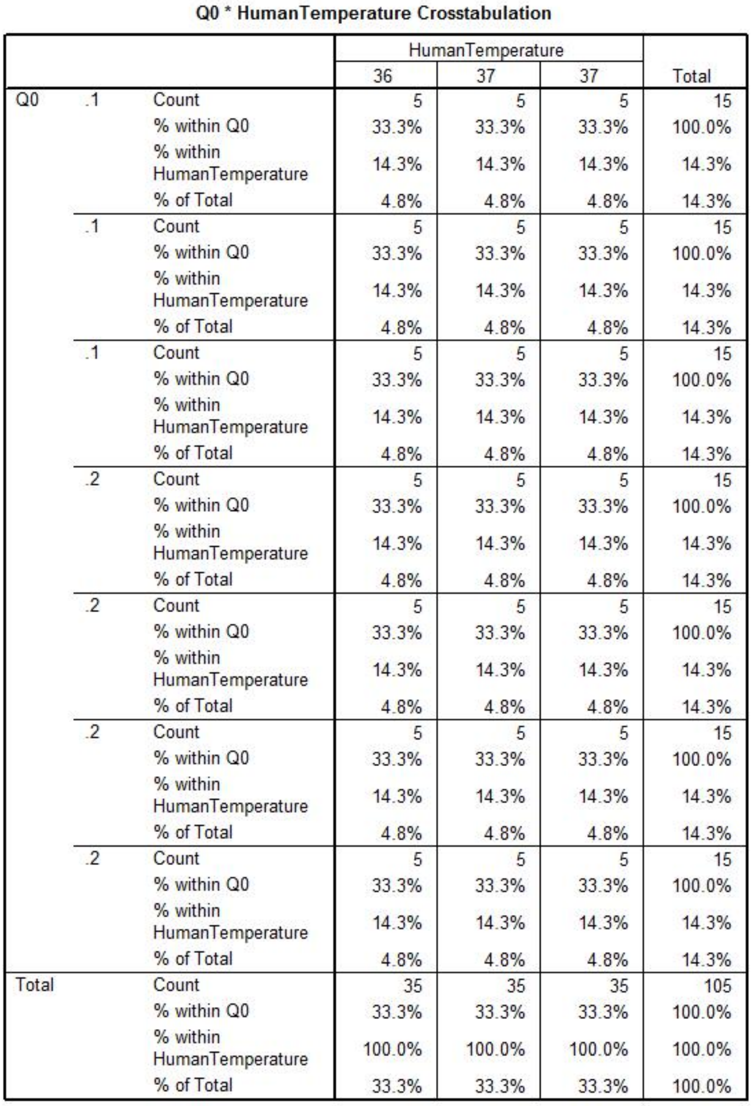


Figure 9: ANOVA bell-shaped curve

From the Picture we know, p=0.073<0.1 and S=0.9965, so the population standard deviation is in a reasonable range. These many experiments results are stable and at the same level, the experiment conclusion has high credibility.

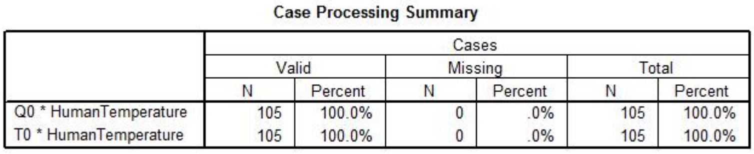
6 Sensitivity Analysis

Some factors inputted in our model may be hard to obtain for there might exist some uncertainty in our inputs. So how will these uncertainty cases affect our model's result? To test the robustness of our model, we employ the Sensitive Analysis to test the relation among the initial factor variable and the result. We use program to simulate heat transfer process. And we get the corresponding water injection solution by using our model. By analysis of the result , we found our model has a good robust behavior. We can use the Crosstabs to realize the local sensitive Analysis.



Q0 * HumanTemperature Crosstabulation

			HumanTemperature			Total
			36	37	37	
Q0	.1	Count	5	5	5	15
		% within Q0	33.3%	33.3%	33.3%	100.0%
		% within HumanTemperature	14.3%	14.3%	14.3%	14.3%
		% of Total	4.8%	4.8%	4.8%	14.3%
	.1	Count	5	5	5	15
		% within Q0	33.3%	33.3%	33.3%	100.0%
		% within HumanTemperature	14.3%	14.3%	14.3%	14.3%
		% of Total	4.8%	4.8%	4.8%	14.3%
	.1	Count	5	5	5	15
		% within Q0	33.3%	33.3%	33.3%	100.0%
		% within HumanTemperature	14.3%	14.3%	14.3%	14.3%
		% of Total	4.8%	4.8%	4.8%	14.3%
	.2	Count	5	5	5	15
		% within Q0	33.3%	33.3%	33.3%	100.0%
		% within HumanTemperature	14.3%	14.3%	14.3%	14.3%
		% of Total	4.8%	4.8%	4.8%	14.3%
	.2	Count	5	5	5	15
		% within Q0	33.3%	33.3%	33.3%	100.0%
		% within HumanTemperature	14.3%	14.3%	14.3%	14.3%
		% of Total	4.8%	4.8%	4.8%	14.3%
	.2	Count	5	5	5	15
		% within Q0	33.3%	33.3%	33.3%	100.0%
		% within HumanTemperature	14.3%	14.3%	14.3%	14.3%
		% of Total	4.8%	4.8%	4.8%	14.3%
	.2	Count	5	5	5	15
		% within Q0	33.3%	33.3%	33.3%	100.0%
		% within HumanTemperature	14.3%	14.3%	14.3%	14.3%
		% of Total	4.8%	4.8%	4.8%	14.3%
Total		Count	35	35	35	105
		% within Q0	33.3%	33.3%	33.3%	100.0%
		% within HumanTemperature	100.0%	100.0%	100.0%	100.0%
		% of Total	33.3%	33.3%	33.3%	100.0%



Case Processing Summary

	Cases					
	Valid		Missing		Total	
	N	Percent	N	Percent	N	Percent
Q0 * HumanTemperature	105	100.0%	0	.0%	105	100.0%
T0 * HumanTemperature	105	100.0%	0	.0%	105	100.0%

From the crosstabs summary, we find that all the data we input is valid, so that the crosstabs test is valid for our model.

7 Conclusion

From the figure, we can tell that with the temperature going up, you have to decrease the rate of water flow to get the same evaluation. Because of difference between the power and water rate, there is fluctuation in the curves. However, if we want to make our strategy practical, we should set a specific temperature. So we set the temperature 48 degrees centigrade at which people will begin to feel scalding.

8 Further Discussion

8.1 Bubble Bath Additive

When we add bubble bath additive into bathtub, bubble generates. The bubbles floating on the water can obviously slow down the water-air heat transfer process. For no exact experiment data is found, we estimate a bubble coefficient and add it in our water-air transfer model to reflect the influence of bubbles. And from the follwing graphic we can clearly see how much do the bubbles influence heat transfer.

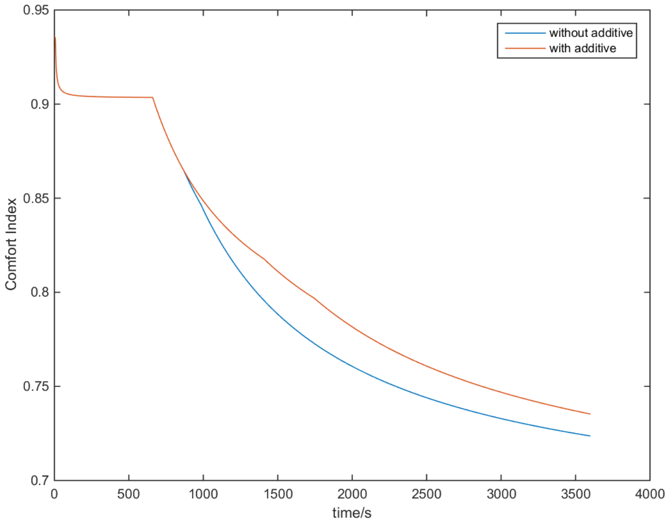


Figure 10: As is shown in the figure, bubble additive increase the comfort index, which instructs us to lower the rate or the temperature of water flow.

9 Non-technical Instructions for Those who Take a Bath

Are you guys ever encountered with the annoying variation of water temperature when taking a bath? The heat transfer including convection and conduction, along with people's motion, changes the water temperature all the time. Sometimes, the changing temperature makes people embarrassed, for the process of maintaining the water temperature is difficult. Hence we have come with up a practical method to cope with that problem.

Our idea is to add a constant trickle of hot water from the faucet to reheat the bath water. If you use a non-spa-style bathtub, you will obviously agree our idea. But you may ask how to set the rate and temperature of the trickle to restrict your water and power rate to a minimum size. Don't worry and here are some tips which can address your problem.

First, fill the bathtub with the water of which temperature is moderate. Then remain the water running and turn up the temperature until the water feels a little bit scalding(about 48 degrees centigrade). At last, adjust the rate of water flow until the water temperature won't noticeably change again. Our mathematical model has proved the temperature is reasonable, which ensure our strategy is useful.

Now you know how to maintain the water temperature in a simple and convenient way. Why not have a go? We hope you can enjoy a cozy and wonderful bath!

10 Strengths and Weaknesses

10.1 Strengths

- **High Accuracy**
  Not only do we separate the bath water into lots of parts to study, but also apply and fit heat transfer formulas on our models. Besides reducing the deviation to a rather small size, those actions accurately help simulate the genuine variation of the water temperature and the movement of water flow.
- **Originative Models**
  We did not establish our models by using other made-up models, but originate our own model by applying physical knowledge and formula.
- **Robustness**
  We implement our strategy under different situations where the air temperature, the material and the external form of the bathtub and the physical condition of the person vary. Finally, we get great effects under those situations.

10.2 Weaknesses

- **Not very Considerate Model**
  Although we take the external form of bathtub into consideration, we fail to notice the internal form of the bathtub should not be always flat, but sometimes curved. Hence our temperature field model isn't applied to those bathtubs with curved internal form.
- **Potential Difference**
  We haven't tried our strategy in person. Correct as our strategy may well be, there may be different in reality.

References

[1] https://en.wikipedia.org/wiki/Convection

[2] https://en.wikipedia.org/wiki/Thermal_conduction